J. Fluid Mech. (1969), *vol.* 35, *part* 4, *pp.* 759–773
Printed in Great Britain

Pathological oscillations of a rotating fluid

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(Received 20 May 1968)

A theoretical study is made of the free periods of oscillation of an incompressible inviscid fluid, bounded by two rigid concentric spheres of radii a, b ($a > b$), and rotating with angular velocity Ω about a common diameter. An attempt is made to use the Longuet-Higgins solution of the Laplace tidal equation as the first term of an expansion in powers of the parameter $\epsilon = (a-b)/(a+b)$, of the solution to the full equations governing oscillations in a spherical shell. This leads to a singularity in the second-order terms at the two critical circles where the characteristic cones of the governing equation touch the shell boundaries.

A boundary-layer type of argument is used to examine the apparent non-uniformity in the neighbourhood of these critical circles, and it is found that, in order to remove the singularity in the pressure, an integrable singularity in the velocity components must be introduced on the characteristic cone which touches the inner spherical boundary. Further integrable singularities are introduced by repeated reflexion at the shell boundaries, and so, even outside the critical region the velocity terms contain what may reasonably be described as a pathological term, generally of order $\epsilon^{\frac{1}{5}}$ compared to that found by Longuet-Higgins, periodic with wavelength $O(\epsilon a)$ in the radial and latitudinal directions.

Some consequences of this result are discussed.

1. Introduction

Although studies on the free periods of oscillation of a rotating incompressible fluid in a container date back to Lord Kelvin (1877) and Greenhill (1880) it is only recently that their properties have received intensive study. The papers just mentioned were largely concerned with containers whose internal shapes were spheroids and an extension to ellipsoids was subsequently carried through by Hough (1898). In more recent times containers whose shapes are circular cylinders with flat ends and including internal boundaries, also coaxial circular cylinders, have been studied by Stewartson (1959). Greenspan, alone (1964, 1965), or in collaboration with Howard (1963) has discussed some general properties of the oscillations, including the effect of a small viscosity. Further an extensive study on more abstract lines has been undertaken in the Soviet Union, an account of which has been given by Rumiantsev (1964, pp. 183–232). Malkus (1967) has also studied the oscillations of a spherical mass of fluid in connexion with the problem of the geomagnetic secular variation, while Roberts & Stewartson (1963) have studied the precessional oscillation of spheroids in connexion with the driving force for the geomagnetic dynamo. Most of the explicit results obtained by these authors depended on it being possible to separate the variables in the governing equation. Another approach has been adopted by Longuet-Higgins (1964, 1965) who considered the container to be the boundaries of a thin spherical shell and was able to determine the free oscillations by consistently neglecting radial motions, so that the governing equation reduced to Laplace's tidal equation. He was also able to generalize his method to include containers whose boundaries form only part of a shell and in a later paper (1968) reports extensive computations. Similar methods have been used by Stewartson (1967) in a related problem in geomagnetism.

In this survey it is of interest to note that success in determining the oscillations depends on being able to use a simplifying argument—separate variables, or neglect radial motions—and that no general explicit results are available. It is natural to enquire what inferences can be made from the information at present available. Can we assume, for example, from these few special cases, that a rotating fluid always has free periods of oscillation, whatever the shape of the container, and is the oscillatory motion always smooth. Physically one might intuitively think so, but from a mathematical standpoint the assumption is dubious. The reason is that the governing equation is hyperbolic, for example

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2} = \lambda^2 \frac{\partial^2 p}{\partial z^2}, \tag{1.1}$$

while the boundary conditions are of the generalized Dirichlet type (a linear relation between p and ∇p on the boundary). This problem is not well-posed in the mathematical sense; we do not know, at present, whether in general solutions exist satisfying the boundary conditions, whether they only exist if λ takes on one of a discrete set of values or whether they only exist if the boundary itself satisfies certain conditions. The particular examples mentioned so far can be regarded as evidence that solutions exist for a discrete set of values of λ. The purpose of the present paper is to provide some evidence that the solutions may not always be smooth.

We were led to this possibility by studying the perturbations of the solutions found by Longuet-Higgins that arise when the thickness of the shell is allowed to increase from zero. Specifically we take the radii of the inner and outer spherical boundaries to be b, a and consider the dependence of the oscillations on

$$\epsilon = \frac{a-b}{a+b}, \tag{1.2}$$

Longuet-Higgins solution being valid when $\epsilon = 0$. The original idea was to expand the velocity components in powers of ϵ. However, we find that the second term in the expansion has a non-integrable singularity on the circles where the characteristic cones of the hyperbolic equation (essentially 1.1) touch the shell. In order to smooth out the singularity as much as possible, the neighbourhood of these circles must be considered by a boundary-layer type of argument. It then emerges that the singularity, in the pressure, can be removed but only at the expense of an integrable singularity in the velocity components on the characteristic cone which touches the *inner* spherical boundary. When this cone meets the outer spherical boundary a new, integrable singularity in the velocity components occurs on the reflected characteristic cone and the pattern is repeated each time the cones meet one or other of the boundaries.

Thus, even outside the neighbourhood of the critical circles, the velocity components contain a pathological term, generally of order $\epsilon^{\frac{1}{2}}$ compared to those found by Longuet-Higgins, periodic with wavelength $O(\epsilon a)$ in the radial and latitudinal directions, and having an integrable singularity on the characteristic cones which, by repeated reflexion, touch the inner boundary.

The precise significance of this result, it must be admitted, is not fully clear to us, handicapped as we are by an almost total absence of precise theorems about badly posed problems. It does seem likely that similar results hold for arbitrary thin shells, topologically equivalent to spherical shells. The essential requirement is that a characteristic cone should touch the inner boundary. There is no contradiction with earlier work because in the only case of an inner boundary studied, two coaxial circular cylinders, no such characteristic cone exists.

As ϵ increases from zero we should expect that the extremely rapid oscillations of the secondary motion, of relative order $\epsilon^{\frac{1}{2}}$, will diminish as the length of the generators of the characteristic cones, interrupted by the shell, increases. The pathology is therefore probably confined to the shells for which $\epsilon \ll 1$. On the other hand there does not appear to be any reason for excluding, as ϵ increases, the integrable singularities on the characteristic cones, which come by reflexion, and it is distinctly possible that they are, in general, features of the free oscillations. Only for containers either without internal boundaries or whose internal boundaries never touch the characteristic cones, and which incidentally have exclusively been studied hitherto, do the singularities disappear.

2. Equations and formal procedure

Consider a shell of incompressible inviscid fluid, bounded by two rigid concentric spheres of radii a, b $(a > b)$, and rotating as if rigid with angular velocity $\mathbf{\Omega}$ about an axis Oz where O is the common centre of the spherical boundaries. A small disturbance is given to this steady motion and we wish to determine the periods of free oscillation of the subsequent motion of the fluid. Denote by $\mathbf{u}$ the fluid velocity measured relative to a set of axes rotating about Oz with angular velocity $\mathbf{\Omega}$ and by (u_R, u_θ, u_ϕ) its components in spherical polar co-ordinates (R, θ, ϕ); here R is the distance of a representative point S from the origin, θ is the angle between OS and Oz, while ϕ is the angle between the planes OSz and a plane through Oz fixed relative to the rotating axes.

The equation of continuity is $\quad \operatorname{div} \mathbf{u} = 0 \qquad (2.1)$

and, neglecting squares and products of $\mathbf{u}$, the equation of momentum reduces to

$$(\partial \mathbf{u}/\partial t) + 2\mathbf{\Omega} \times \mathbf{u} = -\operatorname{grad} \tilde{p}, \tag{2.2}$$

where

$$\tilde{p} = (p/\rho) - \tfrac{1}{2}\Omega^2 R^2 \sin^2\theta, \tag{2.3}$$

p is the pressure and ρ the density of the fluid. It may easily be verified that these equations are separable in ϕ and t: indeed if Q is one of the dependent variables u_R, u_θ, u_ϕ or $\tilde{p}$, then Q may be expressed in the form

$$Q = \Re l q(R,\theta)\, e^{im\phi+i\omega^* t}, \tag{2.4}$$

where m is an integer, which may be either positive or negative, ω^* is a constant to be found and $q(R,\theta)$ is a function of R and θ only. From now on we shall omit the exponential factors and it is understood that the real part is taken. Further let us write

$$\epsilon = \frac{a-b}{a+b}, \quad \mu = \cos\theta, \quad u_R = \frac{\epsilon^2 W}{\sqrt{(1-\mu^2)}}, \quad u_\theta = U, \quad u_\phi = iV,$$

$$\omega^* = \Omega\omega, \quad \tilde{p} = i\Omega R\sin\theta P(R,\mu) \quad \text{and} \quad R = \tfrac{1}{2}(a+b)(1+\epsilon\xi). \tag{2.5}$$

The governing equations then reduce to

$$\frac{\epsilon^3\omega W}{1-\mu^2} - 2\epsilon V = -(1+\epsilon\xi)\frac{\partial P}{\partial\xi} - \epsilon P, \tag{2.6a}$$

$$\omega U - 2\mu V = (1-\mu^2)\frac{\partial P}{\partial\mu} - \mu P, \tag{2.6b}$$

$$\omega V - 2\mu U = -mP + 2\epsilon^2 W \tag{2.6c}$$

and the equation of continuity to

$$\epsilon(1+\epsilon\xi)\frac{\partial W}{\partial\xi} + 2\epsilon^2 W - (1-\mu^2)\frac{\partial U}{\partial\mu} + \mu U - mV = 0. \tag{2.6d}$$

The boundary conditions to be satisfied are that the radial component of velocity vanishes on each of the boundaries, i.e.

$$W = 0 \quad \text{when } \xi = \pm 1. \tag{2.7}$$

Although we shall find it convenient to base our discussion of the properties of the free oscillations on (2·6) it is worth noting that P satisfies the partial differential equation

$$\frac{\partial^2 P}{\partial r^2} + \frac{3}{r}\frac{\partial P}{\partial r} + \frac{1-m^2}{r^2}P = \frac{4-\omega^2}{\omega^2}\frac{\partial^2 P}{\partial z^2}, \tag{2.8a}$$

where $R\cos\theta = z$, $R\sin\theta = r$, while (2.7) may be expressed as

$$(\omega^2 - 4\mu^2)R\frac{\partial P}{\partial R} - 4\mu(1-\mu^2)\frac{\partial P}{\partial\mu} + P(2\omega m + \omega^2) = 0 \tag{2.8b}$$

on $R = a, b$. Greenspan (1964, 1965) has shown that there are no solutions of (2.8a) satisfying (2.8b) for real ω unless $|\omega| \leqslant 2$ and, if $|\omega| < 2$ then the problem is badly posed, as noted in the introduction.

Neglecting radial motions, formally valid in the limit $\epsilon \to 0$, (2.6), (2.7) reduce essentially to Laplace's tidal equations and the periods of the free oscillations have been found by Haurwitz (1940). Our aim here is to extend these by investigating the effect of the terms in (2.6) which depend on ϵ. Intuitively an appropriate expansion to assume is

$$\left.\begin{aligned} U &= U_1(\mu,\xi)+\epsilon U_2(\mu,\xi)+\dots, & V &= V_1(\mu,\xi)+\epsilon V_2(\mu,\xi)+\dots, \\ W &= W_3(\mu,\xi)+\epsilon W_4(\mu,\xi)+\dots, & P &= P_1(\mu,\xi)+\epsilon P_2(\mu,\xi)+\dots, \end{aligned}\right\} \tag{2.9}$$

$$\omega = \omega_1+\epsilon\omega_2+\dots.$$

We are thus attempting to set up an *analytic* expansion procedure with the aim of finding some properties of the free oscillations in shells of finite thickness. Anticipating the result, it is observed however that this is not possible and the expansion must include pathological terms of order $\epsilon^{\frac{1}{2}}$ in U, V.

On substituting (2.9) into (2.6) and comparing coefficients of equal powers of ϵ, we find, from (2.6*a*)

$$\partial P_1/\partial\xi = 0, \tag{2.10}$$

so that P_1 is independent of ξ. It follows immediately that U_1, V_1 are also independent of ξ and hence, from (2.6*d*) that there exists a function $\Psi_1(\mu)$ such that

$$U_1 = m\Psi_1, \quad V_1 = \mu\Psi_1-(1-\mu^2)(d\Psi_1/d\mu). \tag{2.11}$$

Further, from (2.6*c*)

$$mP_1(\mu) = \mu(2m-\omega_1)\Psi_1+\omega_1(1-\mu^2)(d\Psi_1/d\mu) \tag{2.12}$$

and finally, using (2.6*b*)

$$(1-\mu^2)\frac{d^2\Psi_1}{d\mu^2}-4\mu\frac{d\Psi_1}{d\mu}+\Psi_1\left\{\frac{2(m-\omega_1)}{\omega_1}+\frac{1-m^2}{1-\mu^2}\right\} = 0. \tag{2.13}$$

When $m \geqslant 1$, (2.13) is the equation satisfied by Legendre functions or their associated forms and if, on physical grounds we require $P_1(\mu)$ to be bounded as $\mu \to \pm 1$, ω_1 must be equal to one of a discrete set of real values which give us the first approximation to the free periods of oscillation of the fluid. In fact

$$\omega_1 = \frac{2m}{n(n+1)}, \tag{2.14}$$

where n is a positive integer, a result already given by Haurwitz (1940). Otherwise the pressure behaves like $(1-\mu^2)^{-\frac{1}{2}m}$ near $\mu = 1$ and/or $\mu = -1$. When $m = 0$ on the other hand, the only possible solutions of (2.6) are given by

$$\left.\begin{aligned} U_1 &= \frac{\omega_1}{\sqrt{(1-\mu^2)}}, \quad V_1 = \frac{2\mu}{\sqrt{(1-\mu^2)}}, \\ P_1 &= \tfrac{1}{2}(\omega_1^2-4)\frac{\log[(1+\mu)/(1-\mu)]}{\sqrt{(1-\mu^2)}}+\frac{A+4\mu}{\sqrt{(1-\mu^2)}}, \end{aligned}\right\} \tag{2.15}$$

where A is an arbitrary constant. Hence if the pressure is to remain finite at $\mu^2 = 1$, $\omega_1^2 = 4$ from (2.15). The arbitrary constant A can be absorbed into the pressure and then, since $\omega_1^2 = 4$, (2.15) completely satisfies (2.6) for all ϵ. There is therefore, no need to pursue this case further here.†

The solutions of (2.13) with $m > 1$ are clearly similar in structure to those with $m = 1$ and so for the remainder of this paper we shall concentrate on this case; the generalization to arbitrary m may then be carried out in a straightforward way.

Setting $m = 1$, we continue the expansion formally and find, on comparing coefficients of ϵ in (2.6*a*), that

$$P_2(\mu,\xi) = \xi Q_2(\mu) + \overline{P}_1(\mu), \tag{2.16a}$$

where

$$Q_2(\mu) = \mu\omega_1\Upsilon_1 - (2+\omega_1)(1-\mu^2)(d\Upsilon_1/d\mu) \tag{2.16b}$$

and $\overline{P}_1$ is at present an arbitrary function of μ. Continuing we find that

$$U_2(\mu,\xi) = \frac{\xi}{\omega_1^2 - 4\mu^2}\left\{\omega_1(1-\mu^2)\frac{dQ_2}{d\mu} - \mu(\omega_1+2)Q_2\right\} + \overline{U}_1, \tag{2.17a}$$

$$V_2 = \frac{\xi}{\omega_1^2 - 4\mu^2}\left\{2\mu(1-\mu^2)\frac{dQ_2}{d\mu} - (\omega_1+2\mu^2)Q_2\right\} + \overline{V}_1, \tag{2.17b}$$

$$W_3 = \frac{\xi^2-1}{2\xi}\left\{(1-\mu^2)\frac{\partial U_2}{\partial\mu} - \mu U_2 + V_2\right\}. \tag{2.17c}$$

Here $\overline{U}_1$, $\overline{V}_1$, $\overline{P}_1$, satisfy the same relationships as U_1, V_1, P_1 except that the right-hand side of (2.13) is now a function of μ multiplied by ω_2 and it may be established that an acceptable solution of (2.13) can only be found if $\omega_2 = 0$. In that case, however, $\overline{P}_1$ is proportional to P_1 and without loss of generality may be set equal to zero. However, (2.17) does have a disquieting feature which may modify this conclusion but which is beyond the scope of this paper to settle beyond doubt. The feature is that the solutions for U_2, V_2, W_3 all have singularities when $\mu = \pm\frac{1}{2}\omega_1$, and near these circles the second-order terms are not uniformly small with respect to the leading terms of the expansion of the velocity components in powers of ϵ. This singularity is non-integrable and before we can safely make further progress we must, if possible, deal with the apparent non-uniformity near these circles.

† A referee has, however, drawn our attention to a paper by Stern (1963) in which the low frequency axi-symmetrical oscillations are discussed. The limit procedure is more complicated than ours for he supposes that $\omega \to 0$ as $\epsilon \to 0$ so that $\omega/\epsilon^{\frac{1}{2}}$ is bounded. His conclusions are strikingly different in that the oscillations are confined to the immediate neighbourhood of the equator ($\mu = 0$), the stream function of the relative motion decaying inversely with distance from the equator. Whether a reconciliation between Stern's theory and ours is possible, is at present an open question. We note however that he has assumed that the velocity derivatives are bounded everywhere which is in general unjustified in our limit procedure. Further he has assumed that if the stream function decays to zero as $\mu/\epsilon^{\frac{1}{2}} \to \infty$ so do the velocities, which may not be the case.

3. The neighbourhood of the circle $\mu = \mu_0$

When we continue the expansion initiated in the previous section in a formal way we find that

$$P_n = \frac{A_n(\xi, \mu)}{(\omega_1^2 - 4\mu^2)^{2n-5}} \quad (n > 2), \tag{3.1}$$

where A_n is a bounded function of ξ, μ which is not identically zero when $\mu = \pm \frac{1}{2}\omega_1$, and that there are corresponding singularities in U_n, V_n but of one degree higher. It may be inferred that the dominating scaling factor for μ near one of these circles is $\epsilon^{\frac{1}{2}}$ and on writing

$$\mu_0 = \tfrac{1}{2}\omega_1, \quad \mu = \mu_0 + \epsilon^{\frac{1}{2}}\eta, \tag{3.2}$$

it is found that near $\mu = \frac{1}{2}\omega_1$ all these singular terms in P when multiplied by the appropriate power of ϵ are $O(\epsilon^{\frac{3}{2}})$ provided η is finite and non-zero. A similar remark can be made about the neighbourhood of $\mu = -\frac{1}{2}\omega_1$; we shall, however, restrict attention here to the case defined by (3.2). The order of magnitude of the singular terms in U, V, W can also be estimated by a parallel argument and we are led to write, when $\eta = O(1)$

$$U = A_u + \epsilon^{\frac{1}{2}}\bar{u}(\xi, \eta) + \dots, \tag{3.3a}$$

$$V = A_v + \epsilon^{\frac{1}{2}}\bar{v}(\xi, \eta) + \dots, \tag{3.3b}$$

$$W = \epsilon^{-1}\overline{w}(\xi, \eta) + \dots, \tag{3.3c}$$

$$P = A_p + \epsilon^{\frac{1}{2}}\eta B_p + \epsilon(\eta^2 C_p + \xi D_p) + \epsilon^{\frac{3}{2}}\bar{p}(\xi, \eta) + \dots, \tag{3.3d}$$

where

$$\left.\begin{aligned} A_u &= U_1(\mu_0), \quad A_v = V_1(\mu_0), \quad A_p = P_1(\mu_0), \quad B_p = P_1'(\mu_0), \\ C_p &= \tfrac{1}{2}P_1''(\mu_0), \quad D_p = Q_2(\mu_0), \end{aligned}\right\} \tag{3.4}$$

are constant. Further the terms omitted in (3.3) are formally smaller, by a factor $O(\epsilon^{\frac{1}{2}})$ than any of those explicitly mentioned. We now substitute (3.3) into (2.6) and compare coefficients of powers of $\epsilon^{\frac{1}{2}}$, obtaining

$$-2A_v = -D_p - A_p, \tag{3.5a}$$

$$-2\bar{v} = -(\partial\bar{p}/\partial\xi) - \eta B_p \tag{3.5b}$$

from the radial equations (2.6a);

$$\omega_1 A_v - 2\mu_0 A_u = -A_p, \tag{3.6a}$$

$$\omega_1\bar{v} - 2\mu_0\bar{u} - 2\eta A_u = -\eta B_p \tag{3.6b}$$

from (2.6c);

$$\frac{\partial\overline{w}}{\partial\xi} - (1-\mu_0^2)\frac{\partial\bar{u}}{\partial\eta} + \mu_0 A_u - A_v = 0 \tag{3.7}$$

from (2.6d). We could also use (2.6b) but it is more convenient to combine (2.6b) and (2.6c) into

$$(\omega^2 - 4\mu^2)U = -2\mu P + \omega(1-\mu^2)(\partial P/\partial\mu) - \mu\omega P + 4\mu\epsilon^2 W \tag{3.8}$$

which reduces to

$$A_p = (1-\mu_0)B_p, \tag{3.9a}$$

$$4\mu_0 A_u = (1+\mu_0)A_p + \mu_0(1+3\mu_0)B_p - 2\mu_0(1-\mu_0^2)C_p \tag{3.9b}$$

and

$$\begin{aligned} 4\mu_0\eta\bar{u} = {} & \xi D_p\mu_0(1+\mu_0) - \mu_0(1-\mu_0^2)(\partial\bar{p}/\partial\eta) - 2\mu_0\overline{\omega} \\ & + \eta^2[\mu_0(1+\mu_0)C_p + (1+\mu_0)B_p + 4\mu_0^2 C_p + \mu_0 B_p - 2A_u]. \end{aligned} \tag{3.9c}$$

The reduction of these equations involves considerable algebra which is omitted. The final result is obtained after writing

$$\eta = (1-\mu_0^2)^{\frac{1}{2}} x \tag{3.10}$$

and
$$\bar{p} = E_p \eta^3 + \eta \xi F_p + \frac{1}{\sqrt{(1-\mu_0^2)}} \left\{ \frac{1+\mu_0}{2} D_p - \frac{1-\mu_0^2}{2} F_p \right\} \Phi(\xi, x), \tag{3.11}$$

where
$$E_p = \tfrac{1}{6} P_1'''(\mu_0), \quad F_p = Q_2'(\mu_0):$$

then Φ satisfies
$$x \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \xi \, \partial x} = 1 \quad \text{for all} \quad x, |\xi| < 1 \tag{3.12}$$

and
$$2x \frac{\partial \Phi}{\partial \xi} + \frac{\partial \Phi}{\partial x} = 2\xi \quad \text{on} \quad |\xi| = 1, \tag{3.13}$$

while
$$\bar{v} = \frac{1}{2} \frac{\partial \bar{p}}{\partial \xi} + \tfrac{1}{2} \eta B_p, \tag{3.14}$$

$$\bar{u} = \frac{1}{2} \frac{\partial \bar{p}}{\partial \xi} + \frac{\eta}{2\mu_0} [(1+\mu_0) B_p - 2A_u]. \tag{3.15}$$

Provided that Φ remains bounded it follows from (3.10) and (3.3*d*) that as $|x| \to \infty$ the form taken by P is identical with the expansion of P in (2.9) as $\mu \to \mu_0$ to order ϵ. Consequently we must add to (3.13) the condition

$$\Phi \quad \text{is bounded for all} \quad x, \xi; \tag{3.16}$$

then the expansions of P set out in (2.9), (3.3*d*) together with a form analogous to (3.3*d*) but appropriate to the neighbourhood of $\mu = -\mu_0$ altogether constitute a uniformly valid expansion of P to order ϵ.

The asymptotic expansion of a particular integral of (3.12) satisfying (3.13) can be written in the form

$$\Phi_1(x, \xi) = \frac{F_1(\xi)}{x} + \frac{F_2(\xi)}{x^3} + \frac{F_3(\xi)}{x^5} + \dots. \tag{3.17}$$

The functions $F_n(\xi)$ may be determined by substitution into (3.12), (3.13); an interesting feature is that $F_{2n-1}(\xi)$, $F_{2n}(\xi)$ must be determined together. It is noted in passing that if we had continued with our expansion (2.9) a similar phenomenon would have arisen, namely that P_{n-1} $(n > 2)$ cannot be determined fully without reference to P_n. From (3.12) we obtain

$$F_1'' = 1, \quad F_2'' = F_1', \quad F_3'' = 3F_2', \quad \text{etc.}, \tag{3.18}$$

so that
$$\left. \begin{aligned} F_1 &= \tfrac{1}{2}(\xi^2 + b_1 \xi + a_1), \quad F_2 = (\tfrac{1}{6})(\xi^3 + \tfrac{3}{2} b_1 \xi^2 + b_2 \xi + a_2), \\ F_3 &= \tfrac{1}{8}\xi^4 + \tfrac{1}{2} b_1 \xi^3 + \tfrac{1}{4} b_2 \xi^2 + b_3 \xi + a_3, \quad \dots, \end{aligned} \right\} \tag{3.19}$$

where the a, b, c are constants. From (3.13) we have that at $\xi = \pm 1$

$$F_1' = \xi, \quad 2F_2' = F_1, \quad 2F_3' = 3F_2, \quad \dots \tag{3.20}$$

and using (3.19) we have successively

$$b_1 = 0, \quad \tfrac{1}{2}(1 + a_1) = \tfrac{1}{3}(3 + b_2), \quad 2b_3 = \tfrac{1}{6} a_2, \quad b_2 = -\tfrac{1}{6},$$

whence $$a_1 = \tfrac{1}{6}. \tag{3.21}$$

The remaining unknown constant is a_2 and it is in fact arbitrary. However, since we need only one particular integral we can set $a_2 = 0$; in effect we can choose Φ_1 so that F_n is alternatively even and odd in ξ and we have

$$\Phi_1 = \frac{3\xi^2+1}{6x} + \frac{\xi^3-\xi}{6x^3} + \frac{15\xi^4-30\xi^2-1}{120x^5} + \frac{3\xi^5-10\xi^3+7\xi}{24x^7} + \dots; \tag{3.22}$$

as chosen the coefficients of x^{-4n-3}, where n is an integer, vanish on the spherical boundaries $\xi = \pm 1$. The solution is acceptable as $|x| \to \infty$ since it vanishes there, but not as $x \to 0$. Further it is odd in x as would be expected from the governing equations and boundary conditions.

A general solution, acceptable for all x may be found as follows. From (3.12)

$$\Phi = x\xi + F(\tfrac{1}{2}x^2 - \xi) + G(x) \tag{3.23}$$

for some functions F, G to be found. Again, from (3.13)

$$-2xF'(\tfrac{1}{2}x^2-\xi) + 2x^2 + xF'(\tfrac{1}{2}x^2-\xi) + G'(x) + \xi = 2\xi \tag{3.24}$$

on $\xi = \pm 1$; by subtraction the difference equation

$$-xF'(\tfrac{1}{2}x^2-1) + xF'(\tfrac{1}{2}x^2+1) = 2 \tag{3.25}$$

is obtained which may be integrated to give

$$F(\tfrac{1}{2}x^2+1) - F(\tfrac{1}{2}x^2-1) = 2x, \tag{3.26}$$

where we have put the constant of integration equal to zero. The additional function that must otherwise be added to F is even in x and not relevant to our purpose here.

Since Φ is an odd function of x it must vanish on $x = 0$ for all $|\xi| \leqslant 1$ which implies that

$$F(-\xi) = 0 \quad (|\xi| \leqslant 1). \tag{3.27}$$

Hence we have $$F(y) = 0 \quad (-1 \leqslant y \leqslant 1),$$

and from (3.26) $$F(y) = 2\sqrt{[2(y-1)]} \quad (1 \leqslant y \leqslant 3)$$
$$= 2^{\frac{3}{2}}[(y-1)^{\frac{1}{2}} + (y-3)^{\frac{1}{2}}] \quad (3 \leqslant y \leqslant 5).$$

In general therefore $$F(y) = 2^{\frac{3}{2}} \sum_{n=0}^{N-1} (y-2n-1)^{\frac{1}{2}} \tag{3.28}$$

where $|y-2N| \leqslant 1$. The structure of the contribution of $F(\frac{1}{2}x^2-\xi)$ to Φ is illustrated in figure 1. Here $F = 0$ in regions $1\pm$, $F = \pm 2\sqrt{(x^2-2\xi-2)}$ in regions $2\pm$, $3\pm$, $F = \pm 2\sqrt{(x^2-2\xi-2)} \pm 2\sqrt{(x^2-2\xi-6)}$ in regions $4\pm$, $5\pm$, etc. Thus although F is odd in x there is no discontinuity in F anywhere. On the other hand F' is singular as one of the critical curves in figure 1 is approached through decreasing values of $|x|$.

This structure may easily be understood if we refer back for a moment to the basic equation governing the pressure (2.8a). This is hyperbolic and its characteristics in the (r, z)-plane are the straight lines

$$z \pm r\mu_0/\sqrt{(1-\mu_0^2)} = \text{const.} \tag{3.29}$$

At the critical circle $\mu = \mu_0$ one of these characteristics touches the spherical boundaries. If now we convert (3.29) into a relation between x and ξ then one characteristic becomes $\frac{1}{2}x^2 - \xi = \text{const.}$ and the other $x = \text{const.}$ This second family of characteristics is not normal to the spherical boundary: it only appears so because of the scaling relations in (2.5), (3.2). Referring again to figure 1 it is clear that the boundaries of the successive regions are the characteristics of the basic equation (2.8*a*).

The general solution for $F(y)$ may be put into a more compact form by taking the Laplace transform with respect to y, using s as parameter. It then follows from the inversion theorem that

$$F(y) = \frac{1}{2i\sqrt{(2\pi)}}\int_{c-i\infty}^{c+i\infty} \frac{e^{sy}\,ds}{s^{\frac{3}{2}}\sinh s}, \tag{3.30}$$

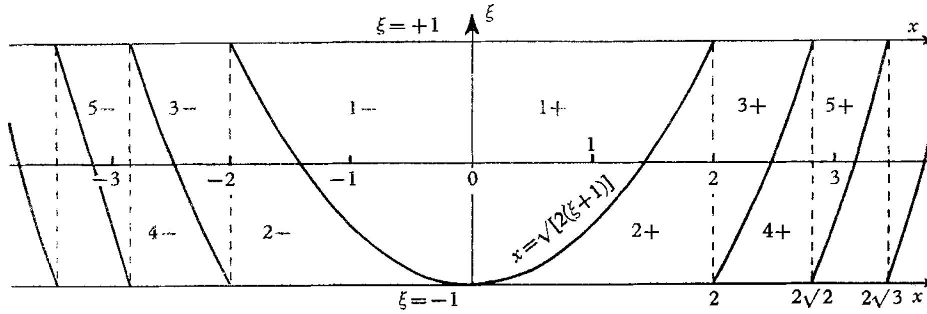

FIGURE 1

where $c > 0$ is real. A formula for $G(x)$ now follows on using (3.24) at $\xi = 1$ and taking, without loss of generality $G(0) = 0$ for

$$G(x) - F(\tfrac{1}{2}x^2 - 1) = x - \tfrac{2}{3}x^3,$$

so that

$$G(x) = x - \frac{2x^3}{3} + \frac{1}{2i\sqrt{(2\pi)}}\int_{c-i\infty}^{c+i\infty} \frac{e^{s(\frac{1}{2}x^2-1)}\,ds}{s^{\frac{3}{2}}\sinh s}. \tag{3.31}$$

An alternative form for $G(x)$ can also be obtained from (3.28) namely

$$G(x) = x - \frac{2x^3}{3} + 2\sum_{n=1}^{N}(x^2-4n)^{\frac{1}{2}}, \tag{3.32}$$

where $4N \leqslant x^2 \leqslant 4(N+1)$. Referring to figure 1, $G - x + \frac{2}{3}x^3$ is zero in regions $1\pm$ and $2\pm$, is equal to $\pm 2\sqrt{(x^2-4)}$ in regions $3\pm$, $4\pm$, etc.

The general solution for Φ now follows from (3.23)

$$\Phi(x,\xi) = x - \frac{2x^3}{3} + x\xi + \frac{1}{2i\sqrt{(2\pi)}}\int_{c-i\infty}^{c+i\infty} \frac{e^{\frac{1}{2}sx^2}\{e^{-s\xi}+e^{-s}\}}{s^{\frac{3}{2}}\sinh s}\,ds. \tag{3.33}$$

When $|x|$ is large, part of this solution should match up with the particular integral found earlier (3.22) and the remainder should be bounded. Now the integral in (3.33) may be regarded as the sum of an integral round a cut along the negative real axis of s and of the residues at the poles of $\sinh s$, other than $s = 0$.

Since these poles are on the imaginary axis the contribution from their residues is finite for all x. The contribution from the cut when $x^2 \gg 1$ is given by

$$\tfrac{1}{2}\sqrt{(2\pi)}\left[\frac{(\frac{1}{2}x^2-\xi)^{\frac{3}{2}}}{(\frac{3}{2})!}+\frac{(\frac{1}{2}x^2-1)^{\frac{3}{2}}}{(\frac{3}{2})!}\right]-\tfrac{1}{12}\sqrt{(2\pi)}\left[\frac{(\frac{1}{2}x^2-\xi)^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\frac{(\frac{1}{2}x^2-1)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\right]+\ldots \tag{3.34}$$

and, when added to $x-\frac{2}{3}x^3+x\xi$ makes a contribution to Φ of

$$\frac{3\xi^2+1}{6x}+O(x^{-3}), \tag{3.35}$$

which is also the leading term of (3.20). So far so good and were it not for the poles we would be able to conclude that the singularity in U_2, V_2, W_3 at the critical circle had been reduced to manageable proportions. Singularities are still present in the velocity components but now have exponent $-\frac{1}{2}$ only and so are integrable.

However, the existence of the poles means that there is another contribution to Φ which is periodic in x^2 and therefore implies a pathological structure to the oscillations as one leaves the neighbourhood of $\mu = \mu_0$. The contribution to (3.30) from the poles of the integrand is

$$K(y) = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n\cos(n\pi y-\frac{3}{4}\pi)}{n^{\frac{3}{2}}}; \tag{3.36}$$

hence the periodic contribution to Φ is equal to

$$K(\tfrac{1}{2}x^2-\xi)+K(\tfrac{1}{2}x^2-1). \tag{3.37}$$

When x^2 is large, this function, although finite, dominates Φ since the remainder tends to zero as $x^2\to\infty$. Further its period $\sim 2/|x|$ when $x^2 \gg 1$ and so the oscillation becomes ever more rapid as x^2 increases. Again since it never dies out, the oscillations persist all over the spherical shell. In physical terms this solution can be associated with disturbances travelling along characteristics bouncing from one spherical boundary to the other. The period is thus the change in μ after moving from one boundary to another along a characteristic of one family and returning on a characteristic of the other, the path being continuous. In terms of ϵ then the period changes from being $O(\epsilon^{\frac{1}{2}})$ near $\mu^2 = \mu_0^2$ to $O(\epsilon)$ elsewhere.

It is of interest to compute the function $K(y)$ explicitly since it dominates Φ when $x^2 \gg 1$. For this purpose the most convenient procedure is to note that when $|y| \leqslant 1$, on the one hand $F(y)$ vanishes since the integral in (3.30) may be completed by the infinite semi-circle to the right of the straight line

$$s = c+it\ (|t| < \infty)$$

and on the other it is equal to

$$K(y)+\frac{1}{2i\sqrt{(2\pi)}}\int_\Gamma \frac{e^{sy}\,ds}{s^{\frac{3}{2}}\sinh s}, \tag{3.38}$$

where Γ is a contour extending to infinity on both sides of the cut, enclosing the origin but no other zeros of $\sinh s$, and with sense such that the origin is on the left of the path of integration. For $-1 \leqslant y \leqslant 0$ we have, since (3.38) is zero,

$$K(y) = -\frac{1}{i\sqrt{(2\pi)}}\int_\Gamma e^{s(y+1)}\,ds\left\{\frac{1}{s^{\frac{3}{2}}(e^{2s}-1)}+\frac{1}{s^{\frac{3}{2}}}\right\}+\frac{1}{i\sqrt{(2\pi)}}\int_\Gamma\frac{e^{s(y+1)}\,ds}{s^{\frac{3}{2}}}$$

and on expanding $e^{s(y+1)}$ term by term as a power series in s in the first integral we obtain

$$K(y) = \left\{2(2y+2)^{\frac{1}{2}} - \sum_{n=0}^{\infty} \frac{(-1)^n (y+1)^n \zeta(n-\frac{1}{2})(n-\frac{3}{2})!}{2^{n-1}\sqrt{\pi}\, n!}\right\}, \tag{3.39}$$

where $\zeta(n)$ is the Riemann zeta function. Although this expansion converges for all $|y| \leqslant 1$ it is more convenient to use another form for K when $0 \leqslant y \leqslant 1$, i.e.

$$K(y) = -\frac{1}{i\sqrt{(2\pi)}}\int_{\Gamma} \frac{ds\, e^{s(y-1)}}{s^{\frac{3}{2}}(1-e^{-2s})}. \tag{3.40}$$

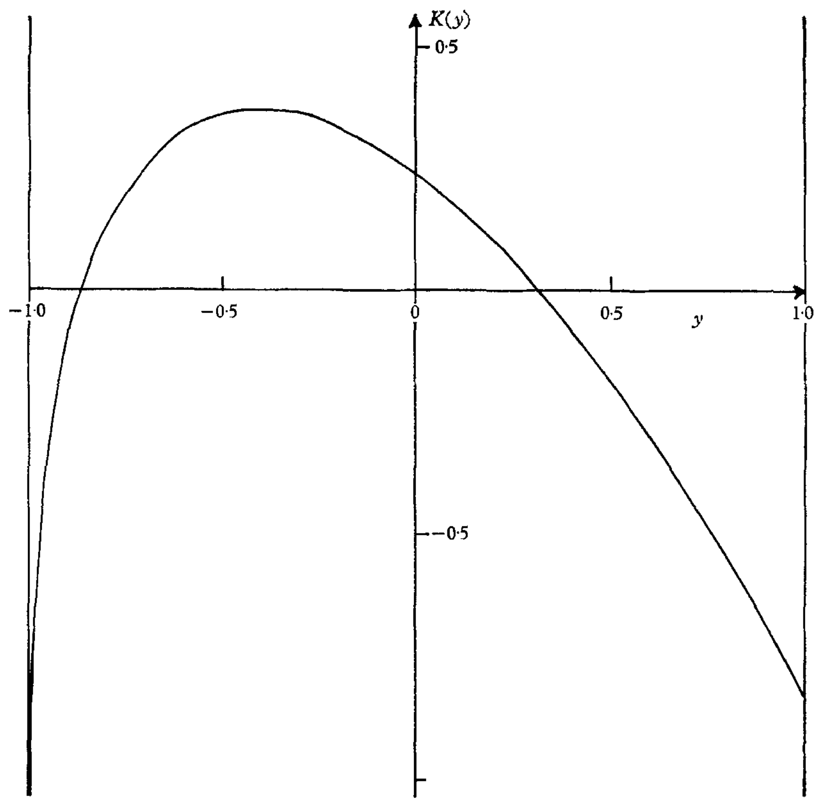

FIGURE 2

On expanding $e^{s(y-1)}$ in powers of $s(y-1)$ and integrating term by term we get

$$K(y) = -\sum_{n=0}^{\infty} \frac{(1-y)^n (n-\frac{3}{2})!\,\zeta(n-\frac{1}{2})}{2^{n-1}\sqrt{\pi}\, n!}. \tag{3.41}$$

It is clear from (3.39), (3.41) that $K(-1) = K(1)$ as required by (3.38). A table of values of $K(y)$ in $|y| \leqslant 1$ is given below, and the function is shown in this range in figure 2.

It is of interest to note that from (3.38) we may write

$$K(y) = 4\zeta(-\tfrac{1}{2}, \tfrac{1}{2}(1+y)),$$

y	-1	$-0{\cdot}9$	$-0{\cdot}8$	$-0{\cdot}7$	$-0{\cdot}6$	$-0{\cdot}5$	$-0{\cdot}4$
$K(y)$	$-0{\cdot}8316$	$-0{\cdot}0865$	$0{\cdot}1286$	$0{\cdot}2512$	$0{\cdot}3232$	$0{\cdot}3612$	$0{\cdot}3733$
y	$-0{\cdot}3$	$-0{\cdot}2$	$-0{\cdot}1$	0	$+0{\cdot}1$	$+0{\cdot}2$	$+0{\cdot}3$
$K(y)$	$0{\cdot}3648$	$0{\cdot}3388$	$0{\cdot}2977$	$0{\cdot}2435$	$0{\cdot}1772$	$0{\cdot}1000$	$0{\cdot}0128$
y	$+0{\cdot}4$	$+0{\cdot}5$	$+0{\cdot}6$	$+0{\cdot}7$	$+0{\cdot}8$	$+0{\cdot}9$	$+1{\cdot}0$
$K(y)$	$-0{\cdot}0838$	$-0{\cdot}1891$	$-0{\cdot}3027$	$-0{\cdot}4241$	$-0{\cdot}5529$	$-0{\cdot}6889$	$-0{\cdot}8316$

TABLE 1

where $\zeta(n, a)$ is the generalized Riemann zeta function (see, for example, Whittaker & Watson 1927, pp. 265–80).

The solution for Φ, whose study we have just completed, satisfies all the boundary conditions. However, during the argument by which it was derived, we assumed (3.26) that a constant of integration vanished and immediately afterwards that $\Phi \equiv 0$ on $x = 0$. If the constant of integration were A instead of zero but $\Phi \equiv 0$ on $x = 0$, then we should have to add a term AN to (3.28) which is not bounded at infinity. Further it is discontinuous on the curved characteristics of figure 1 and this is an unacceptable property for the pressure to have. Hence we must set $A = 0$.

It is not absolutely necessary to have $\Phi \equiv 0$ on $x = 0$ in order to obtain a solution for it is clear from (3.26) that any function of $\frac{1}{2}x^2$ with periodicity 2 may be added to F. Such a function together with a corresponding form for G is bounded at infinity and satisfies the homogeneous equation and boundary conditions for Φ and is therefore not directly forced by the singularities in U_2, V_2 at $\mu = \mu_0$ discussed in § 2. Further, such a contribution to Φ, being even in x, could not eliminate the effect of K both when $x \gg 1$ and when $x \ll -1$. At first sight therefore one might be inclined to exclude such a possibility but as we shall see in the next section it could arise from the periodic residue engendered by the singularity at the critical circle $\mu = -\mu_0$.

4. Discussion

The argument of the previous two sections may be summarized as follows. An attempt is made to use the Longuet-Higgins solution of the Laplace tidal equation as the first term of an expansion, in powers of ϵ, of the solution to the full equations governing oscillations in a spherical shell. It leads, however, to a singularity in the second-order terms at the two critical circles where the characteristic cones touch the shell boundaries. Such a singularity might, in fact, have been anticipated from the form of the governing equation (2.8a) in spherical polars

$$R^2(\mu^2-\mu_0^2)\frac{\partial^2 P}{\partial R^2}+2\mu(1-\mu^2)R\frac{\partial^2 P}{\partial\mu\,\partial R}+(1-\mu^2)(1-\mu^2-\mu_0^2)\frac{\partial^2 P}{\partial\mu^2}$$
$$+(1-\mu^2-4\mu_0^2)R\frac{\partial P}{\partial R}-\mu(3-3\mu^2-4\mu_0^2)\frac{\partial P}{\partial\mu}=0 \quad (4.1)$$

for the coefficient of $\partial^2 P/\partial R^2$ vanishes on each of these circles ($\mu = \pm\mu_0$). In order to remove the non-uniformity at $\mu = \mu_0$, an inner expansion must be set up and this turns out to have a rather surprising character. Above the characteristic cone C_0 which touches the inner boundary ($R = b$ or $\xi = -1$) the solution is smooth, the velocity being given by the Longuet-Higgins formula. On the other side of this cone the pressure develops a new term with an algebraic singularity of exponent $\frac{1}{2}$ at the cone. Further, through the circle where this cone meets the outer boundary ($R = a$ or $\xi = +1$) a characteristic cone C_1 of the other family can be drawn and on the other side of this new cone another singularity of the same kind is added to the pressure. The same process is repeated on all subsequent cones C_n where C_n is the reflexion, at one or other of the boundaries, of C_{n-1}. As the distance from the critical circle increases the pressure is repeatedly augmented in this way, some of which is necessary to form a match with the outer solution. However, the match is not complete as there is a residual periodic term in the inner solution, not accounted for in the outer solution as initially computed. This term, which is $O(\epsilon^{\frac{1}{2}})$ in the tangential velocity components, $O(\epsilon)$ in the radial component and $O(\epsilon^{\frac{3}{2}})$ in the pressure extends therefore into the main body of the fluid where it becomes slowly modified as μ changes but, in essence, repeating the principal features of its structure after two reflexions of the characteristic cones from the boundaries, i.e. after a distance $O(a\epsilon)$ which may be compared with the period $O(a\epsilon^{\frac{1}{2}})$ in the neighbourhood of $\mu = \mu_0$. The radial component also increases from $O(\epsilon)$ to $O(\epsilon^{\frac{1}{2}})$. Another important feature of the structure of this quasi-periodic term is the square root singularity in pressure on one side only of the characteristic cones, and which corresponds to an inverse square root singularity in the velocity components. It is considered that this modification to the earlier elementary solution, in which radial motions are neglected, may fairly be described as pathological. Similar remarks apply to the quasi-periodic solution emanating from $\mu = -\mu_0$.

There are unfortunately very few precise mathematical theorems about the kind of problem under discussion, so that it is possible that the pathology we have just been describing is a creature of the particular limiting process adopted and not actually present when $\epsilon \neq 0$. So far as we can tell it does, nevertheless, seem real enough and in that event the description of free oscillations controlled by an equation like (2.8a) is likely to be complicated. The rapid oscillations in μ we have found become finite in number when $\epsilon > 0$, the total depending on the number of reflexions of characteristic cones needed before the pattern reaches from one pole to the other or repeats itself. The singularity on one side of the characteristic cone, which touches the inner boundary, and on its reflexions is more serious for there seems no reason why these should disappear for $\epsilon > 0$. If present, the implication is that the velocities are no longer integrable in square and the hitherto comforting view that the free oscillations are analytic must be abandoned. In addition there is nothing special about the spherical shell: a similar argument goes through for other shells, provided only that a characteristic cone can touch the inner boundary and so the curious behaviour we have found may well be typical of a large class of rotating cavities containing fluid.

Finally, we note that it has not yet been established that as the pathological wave advances towards $\mu = \pm 1$ from $\mu = \mu_0$ it takes on forms which enable it to pass through the axis of rotation without an unacceptable singularity. The wave needs roughly ϵ^{-1} reflexions to reach the line $\mu = 1$ and the simplest way out of this difficulty is to assume that the actual value of μ_0 (2.14, 3.2) is in error by $O(\epsilon)$ which would give a little freedom near $\mu = 1$ hopefully to enable it to be reached while keeping the pressure finite. Earlier it was established that for a regular expansion of the solution in powers of ϵ, $\omega_2 = 0$ (see (2.9) for definition) which contradicts this suggestion. However, the assumption of a regular expansion is now seen to be untenable, and a further investigation is needed to determine more precisely the dependence of μ_0 on ϵ.

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